THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2060B Mathematical Analysis II
Homework 3 Suggested Solutions

1. (Exercise 6.4.2 of [BS11]) Let $g(x) := |x^3|$ for $x \in \mathbb{R}$. Find $g'(x)$ and $g''(x)$ for $x \in \mathbb{R}$, and $g'''(x)$ for $x \neq 0$. Show that $g'''(0)$ odes not exist.

   Solution. By definition of $g(x)$, we have that $g(x) = x^3$ for $x^3 \geq 0$ and $g(x) = -x^3$ for $x^3 < 0$. Since $x^3 \geq 0$ whenever $x \geq 0$ and $x^3 < 0$ whenever $x < 0$, this means that we can write the definition of g as

   $$g(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$$

   which means

   $$g'(x) = \begin{cases} 3x^2, & x \geq 0 \\ -3x^2, & x < 0 \end{cases} \quad \text{and } g''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0 \end{cases}$$

   We can write $g''(x)$ more succinctly as $g''(x) = 6|x|$. When $x \neq 0$, from what we found for $g''(x)$, we see that

   $$g'''(x) = \begin{cases} 6, & x > 0 \\ -6, & x < 0 \end{cases}$$

   and since $\lim\limits_{x\to 0^+} \frac{g''(x)}{x} = 6$ while $\lim\limits_{x \to 0^-} \frac{g''(x)}{x} = -6$, we see that $g'''(0)$ does not exist. ◀

2. (Exercise 6.4.4 of [BS11]) Show that if $x > 0$, then $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

   Solution. We use Taylor's theroem for $f(x) = \sqrt{1+x}$ at $x_0 = 0$ up to $n = 3$. We have that $f(x_0) = 1$,

   $$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-\frac{1}{2}}, \quad f'(x_0) = \frac{1}{2} \\ f''(x) &= -\frac{1}{4}(1+x)^{-\frac{3}{2}}, \quad f''(x_0) = -\frac{1}{4} \\ f'''(x) &= \frac{3}{8}(1+x)^{-\frac{5}{2}} \end{aligned}$$

   and so we have that

   $$\begin{aligned} f(x) &= 1 + \frac{1}{2}x + R_1(x) = 1 + \frac{1}{2}x - \frac{1}{8}(1+c_1)^{-\frac{3}{2}}x^2 \\ f(x) &= 1 + \frac{1}{2} - \frac{1}{8}x^2 + R_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}(1+c_2)^{-\frac{5}{2}}x^3 \end{aligned}$$

   for some $c_1, c_2 > 0$ and for $x > 0$. Note that for $x > 0$, $R_1(x) < 0$ while $R_2(x) > 0$, so we can conclude that for $x > 0$,

   $$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$$

   as required. ◀

3. (Exercise 6.4.5 of [BS11]) Use the preceding exercise to approximate $\sqrt{1.2}$ and $\sqrt{2}$. What is the best accuracy you can be sure of, using this inequality?

Solution. Using the preceding exercise with $x = 0.2$ for $\sqrt{1.2}$ and $x = 1$ for $\sqrt{2}$, we have

$$1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 \leq \sqrt{1.2} \leq 1 + \frac{1}{2}(0.2) \Rightarrow 1.095 \leq \sqrt{1.2} \leq 1.2$$

and

$$1 + \frac{1}{2}(1) - \frac{1}{8}(1)^2 \leq \sqrt{2} \leq 1 + \frac{1}{2}(1) \Rightarrow 1.375 \leq \sqrt{2} \leq 1.5$$

Since $c_2 > 0$ in the expression for $R_2(x)$, then $(1+c)^{-\frac{5}{2}} < 1$ and so using this inequality, the best accuracy we can obtain is

$$R_3(0.2) \leq \frac{3}{48} \cdot \frac{2}{10} = \frac{1}{80} = 0.0125$$

and

$$R_3(1) \leq \frac{3}{48} \cdot 1 = \frac{1}{16} = 0.0625.$$

◀

4. (Exercise 6.4.9 of [BS11]) If $g(x) := \sin x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$ for each x_0 and x.

Solution. By Taylor's theorem, we have that

$$g(x) = \sum_{k=0}^{n} \frac{g^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x) = \sum_{k=0}^{n} \frac{g^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{g^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

For some c between x and x_0. In any case, since $g^{(n+1)}(x)$ is either $\sin x$ or $\cos x$, we have that $|g^{(n+1)}(c)| \leq 1$, and so we have that

$$|R_n(x)| \leq \left| \frac{g^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \right| \leq \frac{(x-x_0)^{n+1}}{(n+1)!} \to 0 \quad \text{as } n \to +\infty$$

as required. ◀

5. (Exercise 6.4.11 of [BS11]) If $x \in [0,1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

Use this to approximate $\ln 1.5$ with an error less than 0.01. Less than 0.001.

Solution. We have that for $n \in \mathbb{N}$, $\ln^{(n)}(1+x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$, so by Taylor's theorem centred at $x_0 = 0$, we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \frac{n!(1+c)^{-n}}{(n+1)!} x^{n+1}$$

for some c between x and 0. Then since $(1+c)^n \geq 1^n = 1$, we have

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} \right) \right| < \left| (-1)^n \frac{(1+c)^{-n} x^{n+1}}{n+1} \right| \leq \frac{x^{n+1}}{n+1}$$

as required.

For $\ln 1.5$ we are taking $x = \frac{1}{2}$. We require an accuracy up to 10^{-2}, so we require $\frac{1}{n+1} \cdot \frac{1}{2^{n+1}} < 10^{-2} \Leftrightarrow (n+1)2^{n+1} > 10^2$. When $n = 3$,we find that $(n+1)2^{n+1} = 4 \cdot 2^4 = 64$ while for $n = 4$, we have $(n+1)2^{n+1} = 5 \cdot 2^5 = 160 > 10^2$. So we compute the polynomial up to $n = 4$ and obtain $\ln 1.5 \approx 0.40$.

For an accuracy up to 10^{-3}, we require $(n+1)2^{n+1} > 10^3$. We have that when $n = 6, (n+1)2^{n+1} = 7 \cdot 128 = 896$ while for $n = 7, (n+1)2^{n+1} = 8 \cdot 256 = 2048 > 10^3$. So we take $n = 7$ in the polynomial and obtain $\ln 1.5 \approx 0.405$. ◀

6. (Exercise 6.4.15 of [BS11]) Let f be continuous on $[a, b]$ and assume the second derivative f'' exists on (a, b). Suppose that the graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$.

Solution. We use the Mean Value Theorem multiple times. By the Mean Value Theorem applied to f on $[a, x_0]$ and $[x_0, b]$, there is a c_1 between a and x_0 and a c_2 between x_0 and b such that

$$f'(c_1) = \frac{f(x_0) - f(a)}{x_0 - a}, \quad f'(c_2) = \frac{f(b) - f(x_0)}{b - x_0}$$

and since we are on the same line segment joining $(a, f(a))$ to $(b, f(b))$, we have that $\frac{f(x_0) - f(a)}{x_0 - a} = \frac{f(b) - f(x_0)}{b - x_0}$, and so we have that $f'(c_1) = f'(c_2)$. Then using the Mean Value Theorem again, this time applied to f' on $[c_1, c_2]$, we obtain a c_3 in between c_1 and c_2 such that

$$f''(c_3) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = 0$$

as required. ◀

References

[BS11] Robert G. Bartle and Donald R. Sherbert. *Introduction to Real Analysis, Fourth Edition*. Fourth. University of Illinois, Urbana-Champaign: John Wiley & Sons, Inc., 2011. ISBN: 978-0-471-43331-6.